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The initial–boundary value problem for the “good” Boussinesq equation on the bounded domain ☆

Ruying Xue

Department of Mathematics, Zhejiang University, Hangzhou 310027, Zhejiang, PR China

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Abstract

The initial–boundary value problem for the “good” Boussinesq equation on the bounded domain is studied in this article. The local and global well-posedness of this initial–boundary value problem is given.

Keywords: Boussinesq equation; Initial–boundary value problem; Well-posedness

1. Introduction

Considered in this article is the initial–boundary value problem (IBVP henceforth) for the “good” Boussinesq equation on the bounded domain $I = (0, 1)$,

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + \partial_x^4 u + \partial_x^2 u^2 = 0, \quad t > 0,\ x \in I, \\ u(0,t) = \partial_x^2 u(0,t) = 0, \quad u(1,t) = \partial_x^2 u(1,t) = 0, \\ u(x,0) = f(x), \quad \partial_t u(x,0) = \partial_x h(x). \end{cases} \tag{1.1}$$

The equation in (1.1) is known as the “good” Boussinesq equation in comparison with the “bad” Boussinesq equation defined as

$$\partial_t^2 u - \partial_x^2 u - \partial_x^4 u - \partial_x^2 \big(u^2\big) = 0.$$

Equations of Boussinesq-type are a class of essential model equations appearing in physics and fluid mechanics. It is derived by Boussinesq to describe two-dimensional irrotational flows of an inviscid liquid in a uniform rectangular channel. And it also arises in a large range of physical phenomena including the propagation of ion-sound waves in a plasma and nonlinear lattice wave.

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E-mail address: ryxue@zju.edu.cn.

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The study of the pure initial value problem for the "good" Boussinesq equation posed on $\mathbb{R}$,

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + \partial_x^4 u + \partial_x^2 (u^2) = 0, & t > 0,\ x \in \mathbb{R}, \\ u(x,0) = f(x), & \partial_t u(x,0) = \partial_x h(x) \end{cases} \tag{1.2}$$

has recently attracted considerable attention of many mathematicians and physicists. For instance, by using Kato's abstract theory of quasi-linear evolution equation, J. Bona and R. Sachs in [1] proved that the pure initial value problem (1.2) is locally well-posed for smooth data. They proved that for any $f \in H^s(\mathbb{R})$ and $h \in H^{s-1}(\mathbb{R})$ with $s > 5/2$, there exists a time $T > 0$ such that (1.2) has a unique solution $u \in C([0,T]; H^s(\mathbb{R})) \cap C^1([0,T]; H^{s-2}(\mathbb{R})) \cap C^2([0,T]; H^{s-4}(\mathbb{R}))$. In [4], F. Linares established the existence and uniqueness of low regularity solutions to the pure initial-value problem (1.2) when $(f,h) \in L^2(\mathbb{R}) \times H^{-1}(\mathbb{R})$ or $(f,h) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, by using the so-called L^p–L^q smoothing effect of the Strichartz type. Xue in [9] considered the initial value problem for the generalized Boussinesq equation and proved some local or global existence results when the initial data belong to some suitable Besov spaces.

The practical, quantitative use of the Boussinesq-type equations and its relatives does not always involve the pure initial value problem. Instead, the IBVP for the Boussinesq-type equation on a finite domain often comes to the fore. When one considers the IBVP, the difficulty of the evaluation of the contribution of the boundary data occurs. In the literature, very few results are available. By using the Fourier–Bessel series, Varlamov in [8] considered the IBVP of the damped Boussinesq equation in a unit disk, and obtained the global-in-time solvability result. By the finite element Galerkin method, Pani and Saranga considered in [6] the Boussinesq equation with homogenous boundary conditions on a finite domain and proved that IBVP (1.1) possesses a unique weak solution $u \in C([0,T]; H^2(I))$ and $\partial_t u \in C([0,T]; L^2(I))$ provided $f \in H_0^2(I)$ and $h \in H_0^1(I)$. The main result established in the article is:

Let $\frac{1}{2} < s \leqslant \frac{13}{2}$ and let (f,h) belong to the product space $\mathcal{X}^s = H_0^s(I) \times H_0^{s-1}(I)$. Then there exists a positive constant $T_0 > 0$, which depends only on $\|(f,h)\|_{\mathcal{X}^s}$, such that the IBVP (1.1) *possesses a unique solution $u(t,x) \in C([0,T_0]; H^s(I))$. Moreover, the corresponding solution mapping is a Lipschitz correspondence between the initial–boundary data and the solution space.*

Let $1 \leqslant s \leqslant 5$. There exists a positive constant C_0 such that, for $f \in H_0^s(I)$ and $h \in H_0^{s-1}(I)$ with $\|f\|_{H_0^1(I)} + \|h\|_{L^2(I)} \leqslant C_0$, the solution of the IBVP (1.1) *can be extended globally.*

To establish the existence and uniqueness of the solution to the IBVP (1.1) we follow the similar ideas developed in [2] for the study of the Korteweg–de Vries equation posed on a finite domain. In Section 2 we give an explicit representation of the solution to the linear IBVP with vanishing initial data

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + \partial_x^4 u = 0, & t > 0,\ 0 < x < 1, \\ u(0,t) = h_1(t), & \partial_x^2 u(0,t) = \partial_t h_2(t), \\ u(1,t) = g_1(t), & \partial_x^2 u(1,t) = \partial_t g_2(t), \\ u(x,0) = 0, & \partial_t u(x,0) = 0, \end{cases} \tag{1.3}$$

by taking a Laplace transform both sides of the equation in (1.3) and choosing carefully the contour over which the integral is computed. Based on this formula and the smoothing effect present in the pure initial value problem for "good" Boussinesq equation in [4] we obtain some estimates of solutions to some linear IBVPs. Based on these estimates established in Section 2 together with a contraction mapping argument, the local well-posedness for the IBVP (1.1) are obtained in Section 3 in the case $\frac{1}{2} < s \leqslant \frac{13}{2}$. The global existence of the solution to the IBVP (1.1) is considered in Section 4.

In the sequel, we denote by C some large constant which may vary from line to line, and by $\|\cdot\|_X$ the norm in X for a Banach space X, and by $\|(f,g)\|_X = \|f\|_X + \|g\|_X$. Let I be the open interval $(0,1)$ and let $\mathcal{X}^s$ be the product space $H_0^s(I) \times H_0^{s-1}(I)$ with its norm

$$\|(f,h)\|_{\mathcal{X}^s} = \|f\|_{H^s(I)} + \|h\|_{H_0^{s-1}(I)}.$$

The relation $u(y) + iv(y) \sim A(y) + iB(y)$ as $y \to \pm\infty$ means that $\frac{u(y)}{A(y)} \to 1$ and $\frac{v(y)}{B(y)} \to 1$ hold as $y \to \pm\infty$.

2. Linear estimates

In this section we give some smoothing effects for the linear equation associated to (1.1). These estimates will be the main ingredient in the proof of the well-posedness of the solution to the IBVP (1.1). We begin this section by introducing some Sobolev spaces which will be used below.

Let $H^s(\mathbb{R})$ be the set of distributions satisfying $(1+|\xi|)^s\hat{f}(\xi)\in L^2_\xi(\mathbb{R})$, where $\hat{f}(\xi)$ is the Fourier transform of the function $f(x)$. For $s\geqslant 0$ we write

$$H^s(\mathbb{R}^+)=\{f=F|_{\mathbb{R}^+}\colon F\in H^s(\mathbb{R})\},\qquad H^s(I)=\{f=F|_I\colon F\in H^s(\mathbb{R})\}$$

with the norms $\|f\|_{H^s(\mathbb{R}^+)}=\inf\{\|F\|_{H^s(\mathbb{R})}\colon f=F|_{\mathbb{R}^+},\ F\in H^s(\mathbb{R})\}$ and $\|f\|_{H^s(I)}=\inf\{\|F\|_{H^s(\mathbb{R})}\colon f=F|_I,\ F\in H^s(\mathbb{R})\}$, respectively. For $s<0$, we denote by $H^s(\mathbb{R}^+)$ the space of line functionals on $C_0^\infty(\mathbb{R}^+)$ with the norm

$$\|g\|_{H^s(\mathbb{R}^+)}=\sup\{|g(f)|\colon f\in C_0^\infty(\mathbb{R}^+),\ \|f\|_{H_0^{-s}(\mathbb{R}^+)}\leqslant 1\},$$

and denote by $H^s(I)$ the space of line functionals on $C_0^\infty(I)$ with the norm

$$\|g\|_{H^s(I)}=\sup\{|g(f)|\colon f\in C_0^\infty(I),\ \|f\|_{H_0^{-s}(I)}\leqslant 1\}.$$

We define, for $-\infty<s<+\infty$,

$$H_0^s(\mathbb{R}^+)=\{f\in H_0^s(\mathbb{R})\colon \operatorname{supp} f\subseteq[0,+\infty)\}$$

and

$$H_0^s(I)=\{f\in H_0^s(\mathbb{R})\colon \operatorname{supp} f\subseteq[0,1]\}.$$

It is obvious that for $f\in H^s(\mathbb{R})$ we have $f|_{\mathbb{R}^+}\in H^s(\mathbb{R}^+)$ with $\|f|_{\mathbb{R}^+}\|_{H^s(\mathbb{R}^+)}\leqslant\|f\|_{H^s(\mathbb{R})}$, $f|_I\in H^s(I)$ with $\|f|_I\|_{H^s(I)}\leqslant\|f\|_{H^s(\mathbb{R})}$.

Lemma 2.1. *For $s<\frac{1}{2}$ we have $H^s(\mathbb{R}^+)=H_0^s(\mathbb{R}^+)$ and $H^s(I)=H_0^s(I)$. For $k+\frac{1}{2}\leqslant s<k+\frac{3}{2}$ with some nonnegative integer k, we have*

$$H_0^s(\mathbb{R}^+)=\{f\in H^s(\mathbb{R}^+)\colon \operatorname{tr}(\partial^j f)\big|_{x=0}=0,\ j=0,1,\dots,k\},$$

where we set $\operatorname{tr}(\partial^j f)|_{x=0}=\partial^j F(0)$ for $F\in H^s(\mathbb{R})$ with $F|_{\mathbb{R}^+}=f$, and

$$H_0^s(I)=\{f\in H^s(I)\colon \operatorname{tr}(\partial^j f)\big|_{x=0}=0,\ \operatorname{tr}(\partial^j f)\big|_{x=1}=0,\ j=0,1,\dots,k\},$$

where we denote by $\operatorname{tr}(\partial^j f)|_{x=0}=\partial^j F(0)$ and $\operatorname{tr}(\partial^j f)|_{x=1}=\partial^j F(1)$ for $F\in H^s(\mathbb{R})$ with $F|_I=f$.

Proof. The results in the case $s<0$ come from [3], those in the case $s\geqslant 0$ come from Theorems 11.1 and 11.5 in [7]. □

For $\lambda\in\mathbb{C}$ we define the function $\sqrt{\lambda}$ by $\sqrt{\lambda}=e^{\frac{1}{2}(\ln|\lambda|+i\arg\lambda)}$ with $\arg\lambda\in(-\pi,\pi]$, which is an analytic function for $\lambda\notin\mathbb{C}-\overline{\mathbb{R}^+}$. Denote by

$$\gamma_1(\lambda)=\sqrt{\frac{1}{2}-i\sqrt{\lambda^2-\frac{1}{4}}},\qquad \gamma_2(\lambda)=\sqrt{\frac{1}{2}+i\sqrt{\lambda^2-\frac{1}{4}}}.$$

The following lemma comes from a direct computation.

Lemma 2.2.

(1) *For $\operatorname{Re}\lambda>\frac{1}{2}$, $\pm\gamma_1(\lambda)$ and $\pm\gamma_2(\lambda)$ are analytic functions satisfying the equation $\gamma^4-\gamma^2+\lambda^2=0$ with $\operatorname{Re}\gamma_1(\lambda)>0$ and $\operatorname{Re}\gamma_2(\lambda)>0$.*

(2) $\gamma_1(\bar{\lambda}) = \overline{\gamma_2(\lambda)}$ *for* $\operatorname{Re}\lambda > \frac{1}{2}$. *There exist two positive constant* $C_1 < C_2$ *such that for all* $\operatorname{Re}\lambda > \frac{1}{2}$,

$$C_1|\lambda| \leqslant \left|\gamma_1^2(\lambda) - \gamma_2^2(\lambda)\right| \leqslant C_2|\lambda|,$$
$$\left|\gamma_1(\lambda)\right| \leqslant C_2|\lambda|^{\frac{1}{2}}, \qquad \left|\gamma_2(\lambda)\right| \leqslant C_2|\lambda|^{\frac{1}{2}}.$$

(3) *For given* $a \leqslant b$ *and* $\lambda = x + iy$ *with* $\frac{1}{2} < a \leqslant x \leqslant b$ *we have*

$$\gamma_1(\lambda) \sim y^{\frac{1}{2}} - \frac{1}{2}xy^{-\frac{1}{2}}i, \qquad \gamma_2(\lambda) \sim \frac{1}{2}xy^{-\frac{1}{2}} + iy^{\frac{1}{2}} \quad \text{as } y \to +\infty,$$
$$\gamma_1(\lambda) \sim \frac{1}{2}xy^{-\frac{1}{2}} - |y|^{\frac{1}{2}}i, \qquad \gamma_2(\lambda) \sim |y|^{\frac{1}{2}} + \frac{1}{2}x|y|^{-\frac{1}{2}}i \quad \text{as } y \to -\infty,$$
$$\gamma_1^2(\lambda) - \gamma_2^2(\lambda) \sim 2y + 2xi \quad \text{as } y \to +\infty.$$

Let us consider the associated linear IBVP

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + \partial_x^4 u = 0, \quad t > 0,\ 0 < x < 1, \\ u(0,t) = h_1(t), \quad \partial_x^2 u(0,t) = h_2(t), \quad u(1,t) = \partial_x^2 u(1,t) = 0, \\ u(x,0) = 0, \quad \partial_t u(x,0) = 0. \end{cases} \tag{2.1}$$

Lemma 2.3. *For given* $h_1(t), h_2(t) \in C_0^{+\infty}(\mathbb{R}^+)$, *the solution* $W_{1B}(h_1,h_2)(x,t)$ *of the IBVP* (2.1) *has an explicit formula*

$$W_{1B}(h_1,h_2)(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{t(1+iy)}\left[H\big(\tilde{\gamma}_1(y),\tilde{\gamma}_2(y)\big) + H\big(\tilde{\gamma}_2(y),\tilde{\gamma}_1(y)\big)\right]dy,$$

where

$$H(a,b) = \frac{e^{-a}[e^{-a(1-x)} - e^{a(1-x)}]}{(e^{-2a}-1)(b^2-a^2)}\int_0^{+\infty} e^{-(1+iy)\xi}\left[b^2 h_1(\xi) - h_2(\xi)\right]d\xi,$$
$$\tilde{\gamma}_1(y) = \gamma_1(1+iy), \qquad \tilde{\gamma}_2(y) = \gamma_2(1+iy).$$

Proof. For $h(t) \in C_0^{+\infty}(\mathbb{R}^+)$ we define the Laplace transform $\hat{h}(\lambda)$ with $\operatorname{Re}\lambda \geqslant 0$ by

$$\hat{h}(\lambda) = \int_0^{+\infty} e^{-\lambda\xi} h(\xi)\,d\xi.$$

Obviously, for given $k \in \mathcal{N}$ there is a positive constant $C(k,h)$ dependent of k and $h(t)$ such that

$$\left|\hat{h}(\lambda)\right| \leqslant C(k,h)\big(1+|\lambda|^2\big)^{-k}. \tag{2.2}$$

By taking the Laplace transform with respect to t of both sides of the equation in (2.1), the IBVP (2.1) is converted to a one-parameter family of four-order, boundary value problem

$$\begin{cases} \lambda^2\hat{u}(x,\lambda) - \partial_x^2\hat{u}(x,\lambda) + \partial_x^4\hat{u}(x,\lambda) = 0, \quad \operatorname{Re}(\lambda) \geqslant 0,\ 0 < x < 1, \\ \hat{u}(0,\lambda) = \hat{h}_1(\lambda), \quad \partial_x^2\hat{u}(0,\lambda) = \hat{h}_2(\lambda), \quad \hat{u}(1,\lambda) = 0, \quad \partial_x^2\hat{u}(1,\lambda) = 0, \end{cases} \tag{2.3}$$

where λ is the dual variable, $\hat{u}(x,\lambda)$ and $\hat{h}_j(\lambda)$ are the Laplace transforms of $u(x,t)$ and $h_j(t)$ with respect to t, respectively. It is concluded that for any λ with $\operatorname{Re}(\lambda) > 1/2$,

$$\hat{u}(x,\lambda) = \frac{(e^{-\gamma_1(1-x)} - e^{\gamma_1(1-x)})e^{-\gamma_1}}{(e^{-2\gamma_1}-1)(\gamma_2^2-\gamma_1^2)}\left[\gamma_2^2\hat{h}_1(\lambda) - \hat{h}_2(\lambda)\right] + \frac{(e^{-\gamma_2(1-x)} - e^{\gamma_2(1-x)})e^{-\gamma_2}}{(e^{-2\gamma_2}-1)(\gamma_1^2-\gamma_2^2)}\left[\gamma_1^2\hat{h}_1(\lambda) - \hat{h}_2(\lambda)\right]. \tag{2.4}$$

Thus one has the representation, for $0 < x < 1$ and $t > 0$,

$$u(x,t) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} e^{\lambda t}\hat{u}(x,\lambda)\,d\lambda = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{t(1+iy)}\hat{u}(x,1+iy)\,dy$$
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{t(1+iy)}\big[H\big(\tilde{\gamma}_1(y),\tilde{\gamma}_2(y)\big) + H\big(\tilde{\gamma}_2(y),\tilde{\gamma}_1(y)\big)\big]\,dy.$$

The proof is completed. □

Now we consider the following linear IBVP:

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + \partial_x^4 u = 0, \quad t > 0,\ 0 < x < 1, \\ u(0,t) = \partial_x^2 u(0,t) = 0, \quad u(1,t) = g_1(t), \quad \partial_x^2 u(1,t) = g_2(t), \\ u(x,0) = 0, \quad \partial_t u(x,0) = 0. \end{cases} \tag{2.5}$$

Lemma 2.4. *For given $g_1(t), g_2(t) \in C_0^{+\infty}(\mathbb{R}^+)$, the solution $W_{2B}(g_1,g_2)(x,t)$ of IBVP (2.5) has an explicit formula*

$$W_{2B}(g_1,g_2)(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{t(1+iy)}\big[G\big(\tilde{\gamma}_1(y),\tilde{\gamma}_2(y)\big) + G\big(\tilde{\gamma}_2(y),\tilde{\gamma}_1(y)\big)\big]\,dy,$$

where

$$G(a,b) = \frac{e^{-a}[e^{-ax} - e^{ax}]}{(e^{-2a}-1)(b^2-a^2)}\int_0^{+\infty} e^{-(1+iy)\xi}\big[b^2 h_1(\xi) - h_2(\xi)\big]\,d\xi,$$
$$\tilde{\gamma}_1(y) = \gamma_1(1+iy), \qquad \tilde{\gamma}_2(y) = \gamma_2(1+iy).$$

Proof. The proof is similar as that of Lemma 2.3. □

Lemma 2.5. *There exists a positive constant C such that*

$$\bigg\| \int_0^{+\infty} e^{-\tilde{\gamma}_1(y)x} g(y)\,dy \bigg\|_{L_x^2(I)} \leqslant C\big\| |y|^{\frac{1}{4}} g\big\|_{L^2(\mathbb{R}^+)},$$
$$\bigg\| \int_0^{\infty} e^{-\tilde{\gamma}_1(y)(1-x)} g(y)\,dy \bigg\|_{L_x^2(I)} \leqslant C\big\| |y|^{\frac{1}{4}} g\big\|_{L^2(\mathbb{R}^+)}.$$

Proof. One has

$$\bigg\| \int_0^{+\infty} e^{-\tilde{\gamma}_1(y)x} g(y)\,dy \bigg\|_{L_x^2(I)} = 2\bigg\| \int_0^{+\infty} e^{-\tilde{\gamma}_1(s^2)x} s g(s^2)\,ds \bigg\|_{L_x^2(I)}.$$

By Lemma 2.2,

$$\lim_{s\to 0^+} \frac{\operatorname{Re}\tilde{\gamma}_1(s^2)}{s} \geqslant 1, \qquad \lim_{s\to+\infty} \frac{\tilde{\gamma}_1(s^2)}{s} = 1.$$

Using Lemma 2.5 in [2] one deduce

$$\left\| \int_0^{+\infty} e^{-\tilde{\gamma}_1(y)x} g(y)\,dy \right\|_{L_x^2(I)} \leqslant C \left\| s g\big(s^2\big) \right\|_{L_s^2(\mathbb{R}^+)} \leqslant C \left\| |y|^{\frac{1}{4}} g(y) \right\|_{L_y^2(\mathbb{R}^+)}.$$

Similarly, we have

$$\left\| \int_0^{+\infty} e^{-\tilde{\gamma}_1(y)(1-x)} g(y)\,dy \right\|_{L_x^2(I)} \leqslant C \left\| |y|^{\frac{1}{4}} g(y) \right\|_{L_y^2(\mathbb{R}^+)}. \qquad \square$$

Lemma 2.6. *There exists a positive constant C such that*

$$\left\| \int_{-\infty}^{0} e^{-\tilde{\gamma}_1(y)x} g(y)\,dy \right\|_{L_x^2(I)} \leqslant C \left\| |y|^{\frac{1}{4}} g(y) \right\|_{L_y^2(\mathbb{R}^-)},$$

$$\left\| \int_{-\infty}^{0} e^{-\tilde{\gamma}_1(y)(1-x)} g(y)\,dy \right\|_{L_x^2(I)} \leqslant C \left\| |y|^{\frac{1}{4}} g(y) \right\|_{L_y^2(\mathbb{R}^-)}.$$

Proof. It follows from Lemma 2.2 that

$$C_1\big(1+|y|\big)^{-\frac{1}{2}} \leqslant \operatorname{Re}\tilde{\gamma}_1(y) \leqslant C_2 \quad \text{for } y<0.$$

Then we deduce that

$$\begin{aligned}
\left\| \int_{-\infty}^{0} e^{-\tilde{\gamma}_1(y)x} g(y)\,dy \right\|_{L_x^2(I)} &\leqslant C \left\| \int_{-\infty}^{0} e^{|y|^{\frac{1}{2}}xi} g(y)\,dy \right\|_{L_x^2(I)} + C \left\| \int_{-\infty}^{0} \big[e^{-(\tilde{\gamma}_1(y)+|y|^{\frac{1}{2}}i)x} - 1\big] e^{|y|^{\frac{1}{2}}xi} g(y)\,dy \right\|_{L_x^2(I)} \\
&\leqslant C \left\| |y|^{\frac{1}{4}} g(y) \right\|_{L_y^2(\mathbb{R}^-)} + C \left\| \int_{-\infty}^{0} \big|1-e^{-\operatorname{Re}\tilde{\gamma}_1(y)x}\big| \big|g(y)\big|\,dy \right\|_{L_x^2(I)} \\
&\leqslant C \left\| |y|^{\frac{1}{4}} g(y) \right\|_{L_y^2(\mathbb{R}^-)} + C \int_{-\infty}^{0} \big(1+|y|\big)^{-\frac{1}{2}} \big|g(y)\big|\,dy \leqslant C \left\| |y|^{\frac{1}{4}} g(y) \right\|_{L_y^2(\mathbb{R}^-)}.
\end{aligned}$$

The other inequalities in the lemma are proved similarly, their proofs are omitted. $\square$

Let $\phi(\xi) = \xi(1+\xi^2)^{1/2}$ and for $f, h \in \mathcal{S}'(\mathbb{R})$ define

$$V(f)(x,t) = \int_{-\infty}^{+\infty} e^{it\phi(\xi)} e^{ix\xi} \hat{f}(\xi)\,d\xi,$$

$$V_1(f)(x,t) = \frac{1}{2} \int_{-\infty}^{+\infty} \big[e^{it\phi(\xi)+ix\xi} + e^{-it\phi(\xi)+ix\xi}\big] \hat{f}(\xi)\,d\xi$$

and

$$V_2(h)(t,x) = \int_{-\infty}^{+\infty} \big[A(\xi) e^{i(-t\phi(\xi)+x\xi)} + B(\xi) e^{i(t\phi(\xi)+x\xi)}\big]\,d\xi,$$

where $A(\xi) = \frac{\hat{h}(\xi)}{2(1+\xi^2)^{1/2}}$ and $B(\xi) = -\frac{\hat{h}(\xi)}{2(1+\xi^2)^{1/2}}$, $\hat{f}$ and $\hat{h}$ are the Fourier transforms of f and h, respectively. The solution of the initial value problem

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + \partial_x^4 u = 0, & t > 0,\ x \in \mathbb{R}, \\ u(x,0) = f(x), & \partial_t u(x,0) = \partial_x h(x) \end{cases} \tag{2.6}$$

is given by

$$u(x,t) = W_R(f,h)(x,t) = V_1(f)(x,t) + V_2(h)(x,t).$$

The following lemma comes from [4].

Lemma 2.7. *For $s \in (-\infty, +\infty)$, $W_R(f,h)(x,t)$ satisfies*

$$\sup_{t \geqslant 0} \big\| W_R(f,h)(\cdot,t) \big\|_{H^s(\mathbb{R})} \leqslant C \|f\|_{H^s(\mathbb{R})} + C \|h\|_{H^{s-1}(\mathbb{R})}.$$

Let

$$\xi = \xi(\eta) = \frac{\eta}{\sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} + \eta^2}}}$$

be the root of the equation $\phi(\xi) = \eta$ with $\xi \in \mathbb{R}$. We can rewrite the function $V(f)(x,t)$ as

$$V(f)(x,t) = \int_{-\infty}^{+\infty} e^{ix\xi(\eta)} \hat{f}\big(\xi(\eta)\big) \frac{d\xi(\eta)}{d\eta} e^{it\eta}\, d\eta.$$

Thus, for given $\alpha, \beta \in \mathbb{R}$ we have

$$\begin{aligned} \sup_{x \in \mathbb{R}} \big\| |\eta|^\alpha \big(1 + |\eta|^2\big)^{\frac{\beta-\alpha}{2}} \mathcal{F}_t\big[V(f)\big](x,\eta) \big\|_{L^2_\eta(\mathbb{R})} &= \left\| e^{ix\xi(\eta)} \hat{f}\big(\xi(\eta)\big) \frac{d\xi(\eta)}{d\eta} \eta^\alpha (1+\eta)^\beta \right\|_{L^2_\eta(\mathbb{R})} \\ &\leqslant C \left\| \hat{f}\big(\xi(\eta)\big) \frac{d\xi(\eta)}{d\eta} \eta^\alpha (1+\eta)^\beta \right\|_{L^2_\eta(\mathbb{R})} \\ &\leqslant C \left\| \hat{f}(\xi) \left| \frac{d\xi(\eta)}{d\eta} \right|^{1/2} \big|\phi(\xi)\big|^\alpha \big(1 + \big|\phi(\xi)\big|\big)^\beta \right\|_{L^2_\xi(\mathbb{R})} \\ &\leqslant C \big\| \hat{f}(\xi) |\xi|^\alpha \big(1 + |\xi|\big)^{\alpha + 2\beta - \frac{1}{2}} \big\|_{L^2_\xi(\mathbb{R})}. \end{aligned} \tag{2.7}$$

The following lemma comes from the similar argument as that in (2.7).

Lemma 2.8. *For $-\infty < s < +\infty$, there exists a positive constant C such that*

$$\sup_{x \in \mathbb{R}} \big\| W_R(f,h)(x,\cdot) \big\|_{H^{\frac{s}{2}+\frac{1}{4}}(\mathbb{R})} \leqslant C \|f\|_{H^s(\mathbb{R})} + C \|h\|_{H^{s-1}(\mathbb{R})}$$

and

$$\sup_{x \in \mathbb{R}} \big\| \partial_{xx} W_R(f,h)(x,\cdot) \big\|_{H^{\frac{s}{2}-\frac{3}{4}}(\mathbb{R})} \leqslant C \|f\|_{H^s(\mathbb{R})} + C \|h\|_{H^{s-1}(\mathbb{R})}.$$

Lemma 2.9. *Let $s \geqslant 0$ and let*

$$h_1(t) = \int_{-\infty}^{+\infty} e^{-i\phi(\xi)t} \hat{f}(\xi)\, d\xi, \qquad h_2(t) = -\int_{-\infty}^{+\infty} e^{-i\phi(\xi)t} \xi^2 \hat{f}(\xi)\, d\xi,$$

with $h_1^{(k)}(0) = h_2^{(k)}(0) = 0$ either for $k \in \{0, 1, \ldots, [s] - 1\}$ when s is an integer or for $k \in \{0, 1, \ldots, [s]\}$ when s is not an integer. Define by

$$g(y)=\int_0^{+\infty} e^{-(1+iy)t}\big[\tilde{\gamma}_2^2(y)h_1(t)-h_2(t)\big]\,dt.$$

Then one has

$$\big\|\big(1+|y|^2\big)^{\frac{s}{2}}g(y)\big\|_{L_y^2(-\infty,-2)}\leqslant C\|f\|_{H_0^{2s+\frac{1}{2}}(I)}.$$

Proof. It suffices to consider the cases where $s=n$ is an integer and $f\in C_0^\infty(I)$. The analogous result for $s\geqslant 0$ and $f(x)\in H_0^{2s+\frac{1}{2}}(I)$ may be obtained by using the Calderon–Lions interpolation theorem and the dense argument

$$\begin{aligned} y^n g(y) &= \int_0^{+\infty} e^{-iyt}\big[\tilde{\gamma}_2^2(y)D_t^n\big(e^{-t}h_1(t)\big)-D_t^n\big(e^{-t}h_2(t)\big)\big]\,dt \\ &= \sum_{k=0}^n C_k\int_0^{+\infty} e^{-(1+iy)t}\big[\tilde{\gamma}_2^2(y)D_t^k\big(h_1(t)\big)-D_t^k\big(h_2(t)\big)\big]\,dt \\ &= \sum_{k=0}^n C_k\int_{-\infty}^{+\infty} e^{-(1+iy)t}H(t)\big[\tilde{\gamma}_2^2(y)D_t^k\big(h_1(t)\big)-D_t^k\big(h_2(t)\big)\big]\,dt. \end{aligned}$$

Note that

$$\hat{H}(w)=\frac{1}{2}\delta(w)-\frac{1}{2\pi}\frac{i}{w},\qquad D_t^k\big(h_1(t)\big)=\int_{-\infty}^{+\infty} e^{it\eta}\hat{f}_k(\eta)\,d\eta$$

and

$$D_t^k\big(h_2(t)\big)=-\int_{-\infty}^{+\infty} e^{it\eta}\xi^2(\eta)\hat{f}_k(\eta)\,d\eta$$

with $\hat{f}_k(\eta)=\eta^k\hat{f}(\xi(\eta))\frac{d\xi(\eta)}{d\eta}$. Let $z=y-i$ and let

$$g_1(z)=\sum_{k=0}^n C_k\int_{-\infty}^{+\infty} e^{-izt}H(t)\big[\gamma_2^2(iz)D_t^k\big(h_1(t)\big)-D_t^k\big(h_2(t)\big)\big]\,dt.$$

Thus one deduce $y^n g(y)=g_1(y-i)$, $\tilde{\gamma}_2^2(y)=\gamma_2^2(iz)$, $\mathrm{Re}(iz)=1$ for $y<-2$ and

$$g_1(z)=\frac{1}{2}\sum_{k=0}^n C_k\big[\gamma_2^2(iz)+\xi^2(z)\big]\hat{f}_k(z)-\frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{\gamma_2^2(iz)+\xi^2(w)}{z-w}\hat{f}_1(w)\,dw.$$

Notice that $\gamma_2^2(iz)=\frac{1}{2}-\sqrt{z^2+\frac{1}{4}}$ and $\xi^2(w)=-\frac{1}{2}+\sqrt{w^2+\frac{1}{4}}$ for $\mathrm{Re}\,w\geqslant 1$. Then for $y\leqslant -2$,

$$\begin{aligned} g_1(z) &= \frac{i}{2\pi}\sum_{k=0}^n C_k\left[\int_{-\infty}^{+\infty}\left(\frac{z+w}{\sqrt{z^2+\frac{1}{4}}+\sqrt{w^2+\frac{1}{4}}}-1\right)\hat{f}_k(w)\,dw+\int_{-\infty}^{+\infty}\hat{f}_k(w)\,dw\right] \\ &= \frac{i}{2\pi}\sum_{k=0}^n C_k\int_{-\infty}^{+\infty}\left[\frac{y-i+w}{\sqrt{(y-i)^2+\frac{1}{4}}+\sqrt{w^2+\frac{1}{4}}}-1\right]\hat{f}_k(w)\,dw+\frac{i}{2\pi}\sum_{k=0}^n C_k\int_{-\infty}^{+\infty}\big(\widehat{\Delta^{k/2}D^k}f\big)(\xi)\,d\xi \end{aligned}$$

$$= \frac{i}{2\pi} \sum_{k=0}^{n} C_k \int_{-\infty}^{+\infty} K(y,w) \hat{f}_k(w)\,dw + \left(\Delta^{k/2} D^k f\right)(0)$$
$$= \frac{i}{2\pi} \sum_{k=0}^{n} C_k \int_{-\infty}^{+\infty} K(y,w) \hat{f}_k(w)\,dw$$

with $K(y,w) = \varphi_1(y)[\frac{y-i+w}{\sqrt{(y-i)^2+\frac{1}{4}}+\sqrt{w^2+\frac{1}{4}}} - 1]$, because of $(\Delta^{k/2} D^k f)(0) = 0$, where $\varphi_1(y) \in C_0^\infty(\mathbb{R})$ satisfying $\varphi_1(y) = 1$ for $y \leqslant -2$ and $\varphi_1(y) = 0$ for $y \geqslant -1$. It follows from $\operatorname{Re}\sqrt{(y-i)^2 + \frac{1}{4}} \geqslant \frac{1}{2}$ that

$$\sup_y \int_{-\infty}^{\infty} \left|K(y,w)\right|^2 dw < +\infty, \qquad \sup_w \int_{-\infty}^{\infty} \left|K(y,w)\right|^2 dy < +\infty.$$

Then we get

$$\begin{aligned}\left\|y^n g(y)\right\|_{L_y^2(-\infty,-2)} &= \left\|g(y-i)\right\|_{L_y^2(-\infty,-2)} \\ &\leqslant C \sum_{k=0}^{n} C_k \left\| \int_{-\infty}^{+\infty} K(y,w) \hat{f}_k(w)\,dw \right\|_{L_y^2(R)} \leqslant \sum_{k=0}^{n} C_k \|\hat{f}_k\|_{L^2(\mathbb{R})} \leqslant C\|f\|_{H_0^{2n+\frac{1}{2}}(I)}. \qquad \square\end{aligned}$$

Lemma 2.10. *Let $s \geqslant 0$ and*

$$h_1(t) = \int_{-\infty}^{+\infty} e^{-i\phi(\xi)t} \hat{f}(\xi)\,d\xi, \qquad h_2(t) = -\int_{-\infty}^{+\infty} e^{-i\phi(\xi)t} \xi^2 \hat{f}(\xi)\,d\xi,$$

with $h_1^{(k)}(0) = h_2^{(k)}(0) = 0$ either for $k \in \{0, 1, \ldots, [\frac{s}{2} - \frac{1}{4}] - 1\}$ when $[\frac{s}{2} - \frac{1}{4}]$ is an integer or for $k \in \{0, 1, \ldots, [\frac{s}{2} - \frac{1}{4}]\}$ when $[\frac{s}{2} - \frac{1}{4}]$ is not an integer. Then, for any $f \in H_0^s(I)$,

$$\sup_{t \geqslant 0} \left\|e^{-t} W_{1B}(h_1, h_2)(\cdot, t)\right\|_{H^s(I)} \leqslant C\|f\|_{H_0^s(I)}.$$

Proof. It suffices to consider the cases where $s = n$ is an integer and $f \in C_0^\infty(I)$. The analogous result for $s \geqslant \frac{1}{2}$ may be obtained by using the Calderon–Lions interpolation theorem and the dense argument. By Lemma 2.2,

$$\begin{aligned}&\left|e^{-2\tilde{\gamma}_1(y)} - 1\right| \geqslant 1 - e^{-2\operatorname{Re}\tilde{\gamma}_1(y)} \geqslant C\left(1+|y|\right)^{-\frac{1}{2}} \quad \text{for } y \leqslant 0, \\ &\operatorname{Re}\tilde{\gamma}_1(y) \geqslant C > 0 \quad \text{for } y \geqslant 0,\end{aligned}$$

and

$$C_1\left(1+|y|\right)^{\frac{1}{2}} \leqslant \left|\tilde{\gamma}_2(y)\right| \leqslant C_2\left(1+|y|\right)^{\frac{1}{2}}.$$

It follows from Lemmas 2.2, 2.5, 2.6 and 2.9 together with the estimate (2.7) that

$$\begin{aligned}&\left\|e^{-t}\partial_x^n \int_{-\infty}^{+\infty} e^{(1+iy)t} H\left(\tilde{\gamma}_1(y), \tilde{\gamma}_2(y)\right) dy\right\|_{L_x^2(I)} \\ &\quad \leqslant \left\| \int_{0}^{+\infty} e^{iyt} \tilde{\gamma}_1^n(y) H\left(\tilde{\gamma}_1(y), \tilde{\gamma}_2(y)\right) dy \right\|_{L_x^2(I)} + \left\| \int_{-2}^{0} e^{iyt} \tilde{\gamma}_1^n(y) H\left(\tilde{\gamma}_1(y), \tilde{\gamma}_2(y)\right) dy \right\|_{L_x^2(I)}\end{aligned}$$

$$
\begin{aligned}
&+\left\| \int_{-\infty}^{-2} e^{iyt}\tilde{\gamma}_1^n(y) H\big(\tilde{\gamma}_1(y),\tilde{\gamma}_2(y)\big)\,dy \right\|_{L_x^2(I)} \\
&\leqslant C\left\| \frac{\tilde{\gamma}_1^n(y)|y|^{\frac14}}{(e^{-2\tilde{\gamma}_1(y)}-1)(\tilde{\gamma}_2^2(y)-\tilde{\gamma}_1^2(y))} \int_0^{+\infty} e^{-iy\xi}e^{-\xi}\big[\tilde{\gamma}_2^2(y)h_1(\xi)-h_2(\xi)\big]\,d\xi \right\|_{L_y^2(\mathbb{R}^+)} \\
&\quad + C\left\| \frac{\tilde{\gamma}_1^n(y)|y|^{\frac14}}{(e^{-2\tilde{\gamma}_1(y)}-1)(\tilde{\gamma}_2^2(y)-\tilde{\gamma}_1^2(y))} \int_0^{+\infty} e^{-iy\xi}e^{-\xi}\big[\tilde{\gamma}_2^2(y)h_1(\xi)-h_2(\xi)\big]\,d\xi \right\|_{L_y^2(-2,0)} \\
&\quad + C\left\| \frac{\tilde{\gamma}_1^{n+1}(y)|y|^{\frac14}}{(e^{-2\tilde{\gamma}_1(y)}-1)(\tilde{\gamma}_2^2(y)-\tilde{\gamma}_1^2(y))} \int_0^{+\infty} e^{-iy\xi}e^{-\xi}\big[\tilde{\gamma}_2^2(y)h_1(\xi)-h_2(\xi)\big]\,d\xi \right\|_{L_y^2(-\infty,-2)} \\
&\leqslant C\left\| \big(1+|y|\big)^{\frac n2+\frac14} \int_0^{+\infty} e^{-iy\xi}e^{-\xi}h_1(\xi)\,d\xi \right\|_{L_y^2(\mathbb{R})} + C\left\| \big(1+|y|\big)^{\frac n2+\frac34} \int_0^{+\infty} e^{-iy\xi}e^{-\xi}h_2(\xi)\,dy \right\|_{L_y^2(\mathbb{R})} \\
&\quad + C\big\| \big(1+|y|\big)^{\frac n2-\frac14} g(y)\big\|_{L_y^2(\mathbb{R})} \\
&\leqslant C\|f\|_{H_0^n(I)},
\end{aligned}
$$

where $g(y)$ is the function defined in Lemma 2.9. Thus one gets

$$
\sup_{t\geqslant 0}\left\| e^{-t}\int_{-\infty}^{+\infty} e^{(1+iy)t} H\big(\tilde{\gamma}_1(y),\tilde{\gamma}_2(y)\big)\,dy \right\|_{H_x^n(I)} \leqslant C\|f\|_{H_0^n(I)}.
$$

Similarly

$$
\sup_{t\geqslant 0}\left\| e^{-t}\int_{-\infty}^{+\infty} e^{(1+iy)t} H\big(\tilde{\gamma}_2(y),\tilde{\gamma}_1(y)\big)\,dy \right\|_{H_x^n(I)} \leqslant C\|f\|_{H_0^n(I)},
$$

and then

$$
\sup_{t\geqslant 0}\big\| e^{-t} W_{1B}(h_1,h_2)(\cdot,t)\big\|_{H^n(I)} \leqslant C\|f\|_{H_0^n(I)}. \qquad \square
$$

The following lemma comes from the similar argument as that of Lemma 2.10.

Lemma 2.11. *Let $s\geqslant 0$ and*

$$
g_1(t)=\int_{-\infty}^{+\infty} e^{-i\phi(\xi)t}e^{i\xi}\hat{f}(\xi)\,d\xi, \qquad g_2(t)=-\int_{-\infty}^{+\infty} e^{-i\phi(\xi)t}e^{i\xi}\xi^2\hat{f}(\xi)\,d\xi,
$$

with $g_1^{(k)}(0)=g_2^{(k)}(0)=0$ either for $k\in\{0,1,\dots,[\frac s2-\frac14]-1\}$ when $[\frac s2-\frac14]$ is an integer or for $k\in\{0,1,\dots,[\frac s2-\frac14]\}$ when $[\frac s2-\frac14]$ is not an integer. Then one has, for any $f\in H_0^s(I)$,

$$
\sup_{t\geqslant 0}\big\| e^{-t} W_{2B}(g_1,g_2)(\cdot,t)\big\|_{H^s(I)} \leqslant C\|f\|_{H_0^s(I)}.
$$

Proposition 2.1. *Let $T>0$ and $\frac12\leqslant s\leqslant\frac{13}{2}$. For any $f\in H_0^s(I)$ and $h\in H_0^{s-1}(I)$ the following IBVP*

$$
\begin{cases}
\partial_t^2 u-\partial_x^2 u+\partial_x^4 u=0, \quad t>0,\ 0<x<1,\\
u(0,t)=\partial_x^2 u(0,t)=0, \quad u(1,t)=\partial_x^2 u(1,t)=0,\\
u(x,0)=f(x), \quad \partial_t u(x,0)=\partial_x h(x)
\end{cases}
\tag{2.8}
$$

possesses a unique solution $W_C(f,h)(x,t) \in C([0,T]; H^s(I))$ *satisfying*

$$\sup_{0\leqslant t\leqslant T} \big\| W_C(f,h)(\cdot,t)\big\|_{H^s(I)} \leqslant Ce^T\big[\|f\|_{H_0^s(I)} + \|h\|_{H_0^{s-1}(I)}\big].$$

Proof. It is sufficient to consider the case where $f, h \in C_0^\infty(I)$ because of the dense argument. Denote by $(\tilde f(x), \tilde h(x))$ the extension of $(f(x), h(x))$ from I to $\mathbb{R}$ with $(\tilde f(x), \tilde h(x)) = (0,0)$ for $x \notin I$. As $\tilde f \in C_0^\infty(\mathbb{R})$ and $\tilde h \in C_0^\infty(\mathbb{R})$, $W_R(\tilde f, \tilde h)(x,t) \in C^\infty(\mathbb{R}^2)$ is the classical solution to the linear initial value problem

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + \partial_x^4 u = 0, & -\infty < x < +\infty,\ t > 0, \\ u(x,0) = \tilde f(x), & \partial_t u(x,0) = \partial_x \tilde h(x). \end{cases}$$

Denote by

$$\tilde h_1(t) = W_R(\tilde f, \tilde h)(0,t), \qquad \tilde h_2(t) = \big(\partial_x^2 W_R(\tilde f, \tilde h)\big)(0,t)$$

the trace of $W_R(\tilde f, \tilde h)(x,t)$ and $\partial_x^2 W_R(\tilde f, \tilde h)(x,t)$ at $x = 0$. Thus, one deduce

$$\tilde h_1(0) = u(0,0) = \tilde f(0) = 0, \qquad \partial_t \tilde h_1(0) = \partial_x \tilde h(0) = 0,$$
$$\partial_t^2 \tilde h_1(0) = \partial_x^2 \tilde f(0) - \partial_x^4 \tilde f(0) = 0, \qquad \partial_t^3 \tilde h_1(0) = \partial_x^3 \tilde f(0) - \partial_x^5 \tilde f(0) = 0,$$

and similarly,

$$\tilde h_2(0) = \partial_t \tilde h_2(0) = \partial_t^2 \tilde h_2(0) = 0.$$

Lemma 2.10 together with the fact $[\frac{s}{2} - \frac{1}{4}] - 1 \leqslant 2$ implies

$$\sup_{0\leqslant t\leqslant T} \big\| W_{1B}(\tilde h_1, \tilde h_2)\big\|_{H^s(I)} \leqslant Ce^T\big[\|f\|_{H_0^s(I)} + \|h\|_{H_0^{s-1}(I)}\big]. \tag{2.9}$$

Let

$$\tilde g_1(t) = W_R(\tilde f, \tilde h)(1,t), \qquad \tilde g_2(t) = \partial_x^2 W_R(\tilde f, \tilde h)(1,t)$$

be the trace of $W_R(\tilde f, \tilde h)(x,t)$ and $\partial_x^2 W_R(\tilde f, \tilde h)(x,t)$ at $x = 1$, respectively. Replacing Lemma 2.10 with Lemma 2.11 and using the similar argument as (2.9), one get

$$\sup_{0\leqslant t\leqslant T} \big\| W_{2B}(\tilde g_1, \tilde g_2)\big\|_{H^s(I)} \leqslant Ce^T\big[\|f\|_{H_0^s(I)} + \|h\|_{H_0^{s-1}(I)}\big]. \tag{2.10}$$

It follows from Lemma 2.7 that

$$\sup_{t\geqslant 0} \big\| W_R(\tilde f, \tilde h)(\cdot,t)\big\|_{H^s(\mathbb{R})} \leqslant C\big[\|f\|_{H_0^s(I)} + \|h\|_{H_0^{s-1}(I)}\big]. \tag{2.11}$$

Let

$$W_B(\tilde h_1, \tilde h_2, \tilde g_1, \tilde g_2)(x,t) = W_{1B}(\tilde h_1, \tilde h_2)(x,t) + W_{2B}(\tilde g_1, \tilde g_2)(x,t)$$

and

$$W_C(f,h)(x,t) = W_R(\tilde f, \tilde h)(x,t) - W_B(\tilde h_1, \tilde h_2, \tilde g_1, \tilde g_2)(x,t).$$

Then $W_C(f,h)(x,t)$ is well defined for $t \in [0,T]$ and solves the IBVP (2.8) locally. The estimate in the proposition follows from a combination of (2.9) and (2.10) with (2.11). The proof is completed. □

Let $0 \leqslant \varphi(x) \leqslant 1$ be a smooth function defined on $\mathbb{R}$ with $\varphi(x) = 1$ for $|x| \leqslant \frac{1}{8}$ and $\varphi(x) = 0$ for $|x| \geqslant \frac{1}{4}$.

Proposition 2.2. *Let $T > 0$ and $\frac{1}{2} \leqslant s \leqslant \frac{13}{2}$. Assume that $g(x,t) \in L^1([0,T]; H^{s-1}(I))$. Then the IBVP*

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + \partial_x^4 u + \partial_x g(x,t) = 0, & t > 0,\ 0 < x < 1, \\ u(0,t) = \partial_x^2 u(0,t) = 0, \quad u(1,t) = \partial_x^2 u(1,t) = 0, \\ u(x,0) = 0, \quad \partial_t u(x,0) = 0 \end{cases} \tag{2.12}$$

possesses a unique solution $W_I(g)(x,t) \in C([0,T]; H^s(I))$ such that

$$\sup_{0 \leqslant t \leqslant T} \left\| W_I(g)(\cdot,t) \right\|_{H^s(I)} \leqslant Ce^T \int_0^T \left\| g(\cdot,t) \right\|_{H^{s-1}(I)} dt.$$

Proof. The result in the case $\frac{1}{2} \leqslant s < \frac{3}{2}$ comes from Duhamel's principle together with Proposition 2.1, since in this case one has $g(x,\tau) \in H_0^{s-1}(I)$ for $0 \leqslant \tau \leqslant T$ and

$$W_I(g)(x,t) = -\int_0^t W_C\big(0, g(x,\tau)\big)(x, t-\tau)\, d\tau.$$

We only consider the case $\frac{3}{2} \leqslant s < \frac{5}{2}$, the other cases can be considered similarly. Set $\tilde{g}(x,t) = g(x,t) - [g(0,t)\varphi(x) + g(1,t)\varphi(x-1)]$ which belongs to $L^1([0,T]; H_0^{s-1}(I))$ by Lemma 2.1. Consider the linear IBVP

$$\begin{cases} \partial_t^2 v - \partial_x^2 v + \partial_x^4 v + \partial_x \tilde{g}(x,t) = 0, & t > 0,\ 0 < x < 1, \\ v(0,t) = \partial_x^2 v(0,t) = 0, \quad v(1,t) = \partial_x^2 v(1,t) = 0, \\ v(x,0) = 0, \quad \partial_t v(x,0) = 0, \end{cases} \tag{2.13}$$

and the linear IVBP

$$\begin{cases} \partial_t^2 w - \partial_x^2 w + \partial_x^4 w + \partial_x\big(g(0,t)\varphi(x) + g(1,t)\varphi(x-1)\big) = 0, & t > 0,\ 0 < x < 1, \\ w(0,t) = \partial_x^2 w(0,t) = 0, \quad w(1,t) = \partial_x^2 w(1,t) = 0, \\ w(x,0) = 0, \quad \partial_t w(x,0) = 0. \end{cases} \tag{2.14}$$

Duhamel's principle together with Proposition 2.1 implies the existence of the solution v to the IBVP (2.13), which is given by

$$v(x,t) = -\int_0^t W_C\big(0, \tilde{g}(x,\tau)\big)(x, t-\tau)\, d\tau, \quad 0 \leqslant t \leqslant T,$$

and satisfies

$$\sup_{0 \leqslant t \leqslant T} \left\| v(\cdot,t) \right\|_{H^s(I)} \leqslant Ce^T \int_0^T \left\| \tilde{g}(\cdot,\tau) \right\|_{H_0^{s-1}(I)} d\tau \leqslant Ce^T \int_0^T \left\| g(\cdot,\tau) \right\|_{H^{s-1}(I)} d\tau. \tag{2.15}$$

Notice that the equation in the IBVP (2.14) is a linear one. One concludes easily that, by using the classical Galerkin method (see [7]), the IBVP (2.14) possesses a weak solution $w(x,t) \in C([0,T]; H_0^s(I))$ satisfying

$$\sup_{0 \leqslant t \leqslant T} \left\| w(\cdot,t) \right\|_{H^s(I)} \leqslant Ce^T \left\| \big(g(0,t), g(1,t)\big) \right\|_{L_t^1([0,T])} \leqslant Ce^T \int_0^T \left\| g(\cdot,t) \right\|_{H_0^{s-1}(I)} dt. \tag{2.16}$$

Then $u = v + w$ is the solution to the IBVP (2.12) and the estimate follows from a combination of (2.15) with (2.16). $\square$

3. Local well-posedness

In this section we consider the local well-posedness of the IBVP (1.1) in the case $\frac{1}{2} < s \leqslant \frac{13}{2}$. They follow from the estimates put forward in Section 2 together with a contraction mapping argument. Solving (1.1) will be shown to define a continuous mapping from the product space $\mathcal{X}^s = H_0^s(I) \times H_0^{s-1}(I)$ to the space $C([0,T]; H^s(I))$ where the solution $u(x,t)$ resides if $s > \frac{1}{2}$, at least for small values of T. We show that this mapping is Lipschitz.

For $T > 0$ and $\frac{1}{2} < s \leqslant \frac{13}{2}$ we denote by $Z_T^s = C([0,T]; H^s(I))$ with its norm $\|u\|_{Z_T^s} = \sup_{0\leqslant t\leqslant T} \|u(\cdot,t)\|_{H^s(I)}$.

Theorem 3.1. *Let $(f,h) \in H_0^s(I) \times H_0^{s-1}(I)$ and let $\frac{1}{2} < s \leqslant \frac{13}{2}$. Then there exists a positive constant $T_0 > 0$, which depends only on $\|f\|_{H_0^s(I)} + \|h\|_{H_0^{s-1}(I)}$, such that the initial–boundary value problem* (1.1) *possesses a unique solution $u(t,x) = \mathcal{K}(f,h)$ satisfying $u \in C([0,T_0]; H^s(I))$. Moreover, for any $T_1 < T_0$, there exists a neighborhood Y_ϵ centered at (f,h) with its radius $\epsilon > 0$, such that the solution mapping $\mathcal{K}$ is Lipschitz from Y_ϵ to $C([0,T_1]; H^s(I))$.*

Proof. Set

$$X_T^\delta = \{u \in Z_T^s\colon \|u\|_{Z_T^s} \leqslant \delta\},$$

where T and δ are two positive constants to be determined. Define a mapping Φ by

$$\Phi(u) = W_C(f,h)(x,t) + W_I(\partial_x u^2)(x,t).$$

It is sufficient to prove that, for given $(f,h) \in \mathcal{X}^s$, Φ is a contraction mapping from X_T^δ into X_T^δ for suitable $T > 0$ and $\delta > 0$. Denote by $\delta_0 = \|(f,h)\|_{\mathcal{X}^s}$. For $u, v \in X_T^\delta$, it follows from Propositions 2.1 and 2.2 that

$$\begin{aligned}\|\Phi(u)\|_{Z_T^s} &\leqslant \|W_C(f,h)\|_{Z_T^s} + \|W_I(\partial_x u^2)\|_{Z_T^s} \leqslant Ce^T\left[\delta_0 + \int_0^T \|u^2(\cdot,\tau)\|_{H^s(I)}\,d\tau\right] \\ &\leqslant Ce^T\left[\delta_0 + T\|u\|_{Z_T^s}^2\right] \end{aligned}\tag{3.1}$$

and

$$\begin{aligned}\|\Phi(u) - \Phi(v)\|_{Z_T^s} &= \|W_I(\partial_x u^2 - \partial_x v^2)\|_{Z_T^s} \leqslant Ce^T \int_0^T \|u^2(\cdot,\tau) - v^2(\cdot,\tau)\|_{H^s(I)}\,d\tau \\ &\leqslant Ce^T \int_0^T \left(\|u(\tau)\|_{H^s(I)} + \|v(\tau)\|_{H^s(I)}\right)\|u(\tau) - v(\tau)\|_{H^s(I)}\,d\tau \\ &\leqslant CTe^T\left(\|u\|_{Z_T^s} + \|v\|_{Z_T^s}\right)\|u - v\|_{Z_T^s} \leqslant 2CTe^T\delta\|u - v\|_{Z_T^s}. \end{aligned}\tag{3.2}$$

Let $\delta = 2C\delta_0$ and choose $T_0 > 0$ so small that

$$4C^2 T_0 e^{T_0}\delta_0 \leqslant \frac{1}{2}, \qquad e^{T_0}\left(1 + 4C^2 T_0\delta_0\right) \leqslant \frac{3}{2}. \tag{3.3}$$

A combination of (3.1) and (3.2) with (3.3) implies that Φ is a contraction mapping from $X_{T_0}^\delta$ into $X_{T_0}^\delta$, thus we establish the existence and uniqueness of local solution to the IBVP (1.1) in the set $X_{T_0}^\delta$. In fact the uniqueness holds in a large class $C([0,T_0]; H^s(I))$. Suppose $\tilde{u} \in C([0,T_0]; H^s(I))$ satisfying the initial–boundary data, then it is easy to see that for $T' \leqslant T_0$ sufficiently small $\tilde{u} \in X_{T'}^\delta$. Therefore $u = \tilde{u}$ in $[0,T']$. Reapplying this argument we obtain the desired result.

What remains is to prove that for $T_1 < T_0$ the mapping $\mathcal{K}$ from Y_ϵ to $C([0,T_1]; H^s(I))$ is Lipschitz. Let Y_ϵ be the neighborhood at $(f,h) \in \mathcal{X}^s$ with its radius $\epsilon > 0$ to be determined. Let us take u, $\bar{u}$ and $\tilde{u}$ three solutions of (1.1) with initial–boundary data $\mathbf{F} = (f,h) \in Y_\epsilon$, $\bar{\mathbf{F}} = (\bar{f},\bar{h}) \in Y_\epsilon$ and $\tilde{\mathbf{F}} = (\tilde{f},\tilde{h}) \in Y_\epsilon$, respectively. Note that

$$u - \bar{u} = W_C(\mathbf{F} - \bar{\mathbf{F}})(x,t) + W_I\left(\partial_x u^2 - \partial_x \bar{u}^2\right)(x,t).$$

The similar argument as that in (3.2) yields, for $T \leqslant T_1 < T_0$,

$$\begin{aligned}\|u-\tilde{u}\|_{Z^s_T} &\leqslant Ce^T\big[\|\mathbf{F}-\tilde{\mathbf{F}}\|_{\mathcal{X}^s} + T_1\big(\|u\|_{Z^s_T}+\|\tilde{u}\|_{Z^s_T}\big)\|u-\tilde{u}\|_{Z^s_T}\big] \\ &\leqslant Ce^{T_0}\epsilon + \frac{1}{4\delta}\big(2\delta+\|u-\tilde{u}\|_{Z^s_T}\big)\|u-\tilde{u}\|_{Z^s_T},\end{aligned} \tag{3.4}$$

the last inequality holds because of the first inequality in (3.3). Choose $\epsilon > 0$ so small that $Ce^{T_0}\epsilon < \frac{\delta}{16}$. We claim that $\|u-\tilde{u}\|_{Z^s_{T_1}} \leqslant 8C\epsilon$.

In fact, assume that $\|u-\tilde{u}\|_{Z^s_{T'}} \leqslant 8C\epsilon$ holds for some $T' \in [0, T_1]$, we deduce from (3.4) that

$$\|u-\tilde{u}\|_{Z^s_{T'}} \leqslant \frac{3}{4}(8C\epsilon) < 8C\epsilon. \tag{3.5}$$

The claim comes from the continuity method combined with (3.5).

Then, based on the estimate $\|u\|_{Z^s_{T_0}} \leqslant \delta$, we conclude that

$$\|\tilde{u}\|_{Z^s_{T_1}} \leqslant \delta + 8C\epsilon. \tag{3.6}$$

Similarly,

$$\|\bar{u}\|_{Z^s_{T_1}} \leqslant \delta + 8C\epsilon. \tag{3.7}$$

Note that

$$\bar{u}-\tilde{u} = W(\bar{\mathbf{F}}-\tilde{\mathbf{F}})(x,t) + W_I\big(\partial_x\bar{u}^2 - \partial_x\tilde{u}^2\big)(x,t).$$

The similar argument as that in (3.4) together with (3.6) and (3.7) yields

$$\begin{aligned}\|\bar{u}-\tilde{u}\|_{Z^s_{T_1}} &\leqslant Ce^{T_1}\big[\|\bar{\mathbf{F}}-\tilde{\mathbf{F}}\|_{\mathcal{X}^s} + CT_1\big(\|\bar{u}\|_{Z^s_{T_1}}+\|\tilde{u}\|_{Z^s_{T_1}}\big)\|\bar{u}-\tilde{u}\|_{Z^s_{T_1}}\big] \\ &\leqslant Ce^{T_1}\big[\|\bar{\mathbf{F}}-\tilde{\mathbf{F}}\|_{\mathcal{X}^s} + CT_1(2\delta+16C\epsilon)\|\bar{u}-\tilde{u}\|_{Z^s_{T_1}}\big] \\ &\leqslant Ce^{T_0}\|\bar{\mathbf{F}}-\tilde{\mathbf{F}}\|_{\mathcal{X}^s} + \frac{3}{4}\|\bar{u}-\tilde{u}\|_{Z^s_{T_1}},\end{aligned}$$

the last inequality holds because of the first inequality in (3.3). Then we get

$$\|\bar{u}-\tilde{u}\|_{Z^s_{T_1}} \leqslant Ce^{T_0}\|\bar{\mathbf{F}}-\tilde{\mathbf{F}}\|_{\mathcal{X}^s},$$

which means that the solution mapping $\mathcal{K}$ from Y_ϵ to $C([0,T_1];H^s(I))$ is Lipschitz. □

4. Global existence

In this section we consider the global existence results for the IBVP (1.1). The results presented in Section 3 show that the solution to the IBVP (1.1) is locally well-posed in the sense that the length of the time interval $[0, T^*]$ on which the solution exists depends on the quantity $\|f\|_{H^s_0(I)} + \|h\|_{H^{s-1}_0(I)}$. In general, the larger is $\|f\|_{H^s_0(I)} + \|h\|_{H^{s-1}_0(I)}$, the smaller will be T^*. In this section we prove that for the initial data $(f,h) \in H^s_0(I) \times H^{s-1}_0(I)$ with $\|f\|_{H^1_0(I)} + \|h\|_{L^2(I)}$ small, the solution to the IBVP (1.1) can be extended globally.

Lemma 4.1. *Let $T > 0$ be given. Assume $g(x,t) \in L^1([0,T];H^1(I))$ and assume $(f,h) \in H^1_0(I) \times L^2(I)$. Then the linear IBVP with homogenous boundary conditions*

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + \partial_x^4 u + \partial_x^2 g(x,t) = 0, & t > 0,\ 0 < x < 1, \\ u(0,t) = \partial_x^2 u(0,t) = 0, & u(1,t) = \partial_x^2 u(1,t) = 0, \\ u(x,0) = f(x), & \partial_t u(x,0) = \partial_x h(x) \end{cases} \tag{4.1}$$

has a unique solution $u \in C([0,T]; H^1(I))$ satisfying the estimate

$$\int_0^1 \left(\int_0^x \partial_t u(\eta,t)\,d\eta - \int_0^1 (1-\eta)\partial_t u(\eta,t)\,d\eta \right)^2 dx + \left\| u(\cdot,t) \right\|_{H^1(I)} - 2\int_0^t \int_0^1 g(x,\tau)\partial_t u(x,\tau)\,dx\,d\tau$$
$$\leqslant 2\|h\|^2_{L^2(I)} + \|f\|^2_{H^1_0(I)}.$$

Proof. Choose $(f_j, h_j) \in C_0^\infty(I) \times C_0^\infty(I)$ and $g_j(x,t) \in C_0^\infty((0,T); C_0^\infty(I))$ such that, as $j \to +\infty$,

$$\|f - f_j\|_{H^1_0(I)} + \|h - h_j\|_{L^2(I)} \to 0$$

and

$$\left\| g(x,t) - g_j(x,t) \right\|_{L^1([0,T]; H^1(I))} \to 0.$$

Let $u_j(x,t)$ be the solution to Eq. (4.1) with (f,h) and $g(x,t)$ replaced by (f_j,h_j) and $g_j(x,t)$, respectively. The solution $u_j(x,t)$ can be written as

$$u_j(x,t) = W_C(f_j,h_j)(x,t) + W_I(\partial_x g_j)(x,t), \tag{4.2}$$

and satisfies

$$\sup_{0\leqslant t\leqslant T} \left\| u_j(\cdot,t) - u_k(\cdot,t) \right\|_{H^1(I)} \to 0$$

as $j,k \to +\infty$, because of Propositions 2.1 and 2.2. Using $g_j(0,t) = g_j(1,t) = 0$, $u_j(0,t) = \partial_x^2 u_j(0,t) = 0$ and $u_j(1,t) = \partial_x^2 u_j(1,t) = 0$, integrating both sides of the equation

$$\partial_t^2 u_j - \partial_x^2\big(u_j - \partial_x^2 u_j - g_j(x,t)\big) = 0$$

with respect to x, one obtains

$$\int_0^x (x-\eta)\partial_t^2 u_j(\eta,t)\,d\eta\,d\xi - x\int_0^1 (1-\eta)\partial_t^2 u_j(\eta,t)\,d\eta = u_j(x,t) - \partial_x^2 u_j(x,t) - g_j(x,t). \tag{4.3}$$

Integrating by parts yields

$$\int_0^1 \left(\int_0^x (x-\eta)\partial_t^2 u_j(\eta,t)\,d\eta - x\int_0^1 (1-\eta)\partial_t^2 u_j(\eta,t)\,d\eta \right) \partial_t u(x,t)\,dx$$
$$= -\int_0^1 \left(\int_0^x \partial_t u_j(\eta,t)\,d\eta \right)\left(\int_0^x \partial_t^2 u_j(\eta,t)\,d\eta \right) dx + \int_0^1 (1-\eta)\partial_t^2 u_j(\eta,t)\,d\eta \int_0^1 \left(\int_0^x \partial_t u(\eta,t)\,d\eta \right) dx$$
$$= -\int_0^1 \left(\int_0^x \partial_t u_j(\eta,t)\,d\eta - \int_0^1 (1-\eta)\partial_t u_j(\eta,t)\,d\eta \right)\left(\int_0^x \partial_t^2 u_j(\eta,t)\,d\eta - \int_0^1 (1-\eta)\partial_t^2 u_j(\eta,t)\,d\eta \right) dx$$
$$= -\frac{1}{2}\frac{d}{dt}\int_0^1 \left(\int_0^x \partial_t u_j(\eta,t)\,d\eta - \int_0^1 (1-\eta)\partial_t u_j(\eta,t)\,d\eta \right)^2 dx.$$

Multiplying both sides of Eq. (4.3) by $\partial_t u_j$ and then integrating with respect to x yields

$$\frac{1}{2}\frac{d}{dt}\left[\int_0^1\left(\int_0^x \partial_t u_j(\eta,t)\,d\eta-\int_0^1(1-\eta)\partial_t u_j(\eta,t)\,d\eta\right)^2dx+\left\|u_j(\cdot,t)\right\|_{H^1(I)}^2\right]$$
$$=\int_0^1 g_j(x,t)\partial_t u_j(x,t)\,dx. \tag{4.4}$$

Integrating both sides of Eq. (4.4) with respect to t, one arrives at

$$\int_0^1\left(\int_0^x \partial_t u_j(\eta,t)\,d\eta-\int_0^1(1-\eta)\partial_t u_j(\eta,t)\,d\eta\right)^2dx+\left\|u_j(\cdot,t)\right\|_{H^1(I)}^2-2\int_0^t\int_0^1 g_j(x,\tau)\partial_t u_j(x,\tau)\,dx\,d\tau$$
$$\leqslant 2\|h_j\|_{L^2(I)}^2+\|f_j\|_{H_0^1(I)}^2. \tag{4.5}$$

Denote by $u(x,t)$ the limit of $\{u_j(x,t)\}_{j=1}^{+\infty}$ in the space $C([0,T];H^1(I))$, let $j\to+\infty$ in (4.2) and in (4.5), we deduce that

$$u(x,t)=W_C(f,h)(x,t)+W_I(\partial_x g)(x,t),$$

which means that $u\in C([0,T];H^1(I))$ is a solution to the IBVP (4.1), and that

$$\int_0^1\left(\int_0^x \partial_t u(\eta,t)\,d\eta-\int_0^1(1-\eta)\partial_t u(\eta,t)\,d\eta\right)^2dx+\left\|u(\cdot,t)\right\|_{H^1(I)}^2-2\int_0^t\int_0^1 g(x,\tau)\partial_t u(x,\tau)\,dx\,d\tau$$
$$\leqslant 2\|h\|_{L^2(I)}^2+\|f\|_{H_0^1(I)}^2.$$

The proof is completed. □

Theorem 4.1. *There exists a positive constant C_0 such that for initial data $(f,h)\in H_0^1(I)\times L^2(I)$ with $\|f\|_{H_0^1(I)}+\|h\|_{L^2(I)}\leqslant C_0$, the IBVP* (1.1) *has a unique global solution $u\in C([0,\infty);H^1(I))$ satisfying*

$$\sup_{t\geqslant 0}\left\|u(\cdot,t)\right\|_{H^1(I)}\leqslant C\left(\|f\|_{H_0^1(I)}+\|h\|_{L^2(I)}\right),$$
$$\sup_{t\geqslant 0}\left\|\int_0^x \partial_t u(\eta,t)\,d\eta-\int_0^1(1-\eta)\partial_t u(\eta,t)\,d\eta\right\|_{L_x^2(I)}\leqslant C\left(\|f\|_{H_0^1(I)}+\|h\|_{L^2(I)}\right).$$

Proof. The local well-posedness presented in Section 3, combined with Lemma 4.1, means that there exists a positive constant T depending only on $\|f\|_{H_0^1(I)}+\|h\|_{L^2(I)}$ such that, the IBVP (1.1) has a unique solution $u\in C([0,T];H^1(I))$. Based on this local well-posedness, it is sufficient to establish a priori estimates

$$\sup_{0\leqslant t\leqslant T}\left\|u(\cdot,t)\right\|_{H^1(I)}\leqslant C\left(\|f\|_{H_0^1(I)}+\|h\|_{L^2(I)}\right) \tag{4.6}$$

and

$$\sup_{0\leqslant t\leqslant T}\left\|\int_0^x \partial_t u(\eta,t)\,d\eta-\int_0^1(1-\eta)\partial_t u(\eta,t)\,d\eta\right\|_{L_x^2(I)}\leqslant C\left(\|f\|_{H_0^1(I)}+\|h\|_{L^2(I)}\right), \tag{4.7}$$

with a positive constant C independent of T, since (4.6) and (4.7) imply that, if one denotes by

$$\tilde f(x)=u(x,T)\quad\text{and}\quad \tilde h(x)=\left[\int_0^x \partial_t u(\eta,t)\,d\eta-\int_0^1(1-\eta)\partial_t u(\eta,t)\,d\eta\right]_{t=T},$$

$u|_{t=T} = \tilde{f}(x)$ and $u_t|_{t=T} = \partial_x \tilde{h}(x)$ satisfying $(\tilde{f}, \tilde{h}) \in H_0^1(I) \times L^2(I)$ with

$$\|\tilde{f}\|_{H_0^1(I)} + \|\tilde{h}\|_{L^2(I)} \leqslant C\big(\|h\|_{H_0^1(I)} + \|h\|_{L^2(I)}\big).$$

Let $g(x,t) = u^2(x,t)$, which satisfies $g(x,t) \in C([0,T]; H^1(I))$. The estimate stated in Lemma 4.1 yields

$$\begin{aligned} &\left\| \int_0^x \partial_t u(\eta,t)\,d\eta - \int_0^1 (1-\eta)\partial_t u(\eta,t)\,d\eta \right\|_{L_x^2(I)} + \big\|u(\cdot,t)\big\|_{H^1(I)}^2 + \frac{2}{3}\int_0^1 u^3(x,t)\,dx \\ &\quad \leqslant 2\|h\|_{L^2(I)}^2 + \|f\|_{H_0^1(I)}^2 + \frac{2}{3}\int_0^1 f^3(x)\,dx \leqslant 2\|h\|_{L^2(I)}^2 + \|f\|_{H^1(I)}^2 + C_1\|f\|_{H^1(I)}^3, \end{aligned} \tag{4.8}$$

where $C_1 = \frac{2}{3}\sup\{\|f\|_{L^3(I)}^3\colon f \in H_0^1(I),\ \|f\|_{H_0^1(I)} = 1\} < +\infty$. Denote by

$$K_0 = 2\|h\|_{L^2(I)}^2 + \|f\|_{H_0^1(I)}^2 + C_1\|f\|_{H_0^1(I)}^3.$$

It follows from (4.8) that

$$\big\|u(\cdot,t)\big\|_{H^1(I)}^2 - C_1\big\|u(\cdot,t)\big\|_{H^1(I)}^3 \leqslant K_0, \quad t \in [0,T]. \tag{4.9}$$

Set $X(t) = \|u(\cdot,t)\|_{H^1(I)}^2$, and rewrite (4.9) as

$$X(t) - C_1 X(t)^{\frac{3}{2}} \leqslant K_0, \quad t \in [0,T]. \tag{4.10}$$

Note that, for $K_0 < \frac{4}{27}C_1^{-2}$, the inequality

$$X - C_1 X^{\frac{3}{2}} \leqslant K_0$$

is satisfied if and only if $X \in [0,\beta_1] \cup [\beta_2, +\infty)$ for positive constants β_1 and β_2 satisfying $0 < \beta_1 < \frac{4}{9}C_1^{-2} < \beta_2 < +\infty$ and $\beta_1 \leqslant C K_0^{\frac{1}{2}}$. Choose $C_0 < \frac{2}{9}C_1^{-2}$ so small that $2C_0^2 + C_1C_0^3 < \frac{4}{27}C_1^{-2}$. For $(f,h) \in H_0^1(I) \times L^2(I)$ with $\|f\|_{H_0^1(I)} + \|h\|_{L^2(I)} \leqslant C_0$, we have $K_0 < \frac{4}{27}C_1^{-2}$, $X(0) = \|u(\cdot,0)\|_{H^1(I)} < \frac{4}{9}C_1^{-2}$ and

$$X(0) - C_1 X(0)^{\frac{3}{2}} \leqslant K_0.$$

Then $X(0) \in [0,\beta_1]$, and the continuity of $X(t)$ combined with (4.10) allows us to conclude that $X(t) \in [0,\beta_1]$ for all $t \in [0,T]$. Thus $\sup_{0\leqslant t\leqslant T}\|u(\cdot,t)\|_{H^1(I)} \leqslant \beta_1 \leqslant CK_0^{\frac{1}{2}}$, where it implies the first inequality in the theorem. The second inequality in the theorem comes from a combination of (4.8) with the inequality $\sup_{0\leqslant t\leqslant T}\|u(\cdot,t)\|_{H^1(I)} \leqslant CK_0^{\frac{1}{2}}$. □

Lemma 4.2. *Let $(f_j,h_j) \in H_0^1(I) \times L^2(I)$ satisfy $\|f_j\|_{H_0^1(I)} + \|h_j\|_{L^2(I)} \leqslant C_0$ with the constant C_0 obtained in Theorem 4.1 for $j = 1,2$, and let u_j be the solution of the IBVP (1.1) associated with the initial data (f_j,h_j), respectively. For given $T > 0$, there exists a T-dependent and non-decreasing function $A_T(\xi)$ from $[0,+\infty)$ to $\mathbb{R}^+$ such that*

$$\sup_{0\leqslant t\leqslant T}\big\|u_1(\cdot,t) - u_2(\cdot,t)\big\|_{H^1(I)} \leqslant A_T\left(\sum_{j=1}^2\big(\|f_j\|_{H_0^1(I)} + \|h_j\|_{L^2(I)}\big)\right)\big(\|f_1 - f_2\|_{H_0^1(I)} + \|h_1 - h_2\|_{L^2(I)}\big).$$

Proof. Denote by $v = u_1 - u_2$. It is seen that $v \in C([0,+\infty); H^1(I))$ solves the IBVP

$$\begin{cases} \partial_t^2 v - \partial_x^2 v + \partial_x^4 v + \partial_x^2\big((u_1+u_2)v\big) = 0, & t > 0,\ x > 0, \\ v(0,t) = \partial_x^2 v(0,t) = 0, \quad v(1,t) = \partial_x^2 v(1,t) = 0, \\ v(x,0) = f_1(x) - f_2(x), \quad \partial_t v(x,0) = \partial_x\big(h_1(x) - h_2(x)\big). \end{cases} \tag{4.11}$$

Write

$$v(x,t) = W_C(f_1 - f_2, h_1 - h_2)(x,t) + W_I\big(\partial_x\big((u_1+u_2)v\big)\big)(x,t).$$

It follows from Propositions 2.1 and 2.2 that, for $0 \leqslant t \leqslant T$,

$$\begin{aligned}
\big\|v(\cdot,t)\big\|_{H^1(I)} &\leqslant Ce^T\big(\|f_1-f_2\|_{H_0^1(I)} + \|h_1-h_2\|_{L^2(I)}\big) + Ce^T\int_0^t \big\|\big(u_1(\cdot,\tau)+u_2(\cdot,\tau)\big)v(\cdot,\tau)\big\|_{H^1(I)}\,d\tau \\
&\leqslant Ce^T\big(\|f_1-f_2\|_{H_0^1(I)} + \|h_1-h_2\|_{L^2(I)}\big) \\
&\quad + Ce^T\sup_{0\leqslant t\leqslant T}\big(\big\|u_1(\cdot,t)\big\|_{H^1(I)} + \big\|u_2(\cdot,t)\big\|_{H^1(I)}\big)\int_0^t \big\|v(\cdot,\tau)\big\|_{H^1(I)}\,d\tau. \qquad (4.12)
\end{aligned}$$

Then the lemma follows from (4.12) and Theorem 4.1 combined with the Gronwall inequality. We complete the proof. □

Lemma 4.3. *For given $T > 0$, there exists a T-dependent and non-decreasing function $B_T(\xi)$ from $[0,+\infty)$ into $\mathbb{R}^+$ such that, for any $(f,h) \in H_0^5(I) \times H_0^4(I)$ with $\|f\|_{H_0^1(I)} + \|h\|_{L^2(I)} \leqslant C_0$, the IBVP with homogenous boundary conditions* (1.1) *has a unique solution $u \in C([0,T]; H^5(I))$. Moreover, we have*

$$\sup_{0\leqslant t\leqslant T}\big\|u(\cdot,t)\big\|_{H^5(I)} \leqslant B_T\big(\|f\|_{H_0^1(I)} + \|h\|_{L^2(I)}\big)\big(\|f\|_{H_0^5(I)} + \|h\|_{H_0^4(I)}\big). \qquad (4.13)$$

Proof. Let $v_0(x,t) = u(x,t)$, $v_1(x,t) = \partial_t u(x,t)$ and $v_2(x,t) = \partial_t^2 u(x,t)$. Denote by $f_0(x) = f(x)$, $h_0(x) = h(x)$, $f_1(x) = \partial_x h(x)$, $h_1(x) = \partial_x f(x) - \partial_x^3 f(x) - \partial_x(f^2(x))$, $f_2(x) = \partial_x h_1(x)$ and $h_2(x) = \partial_x f_1(x) - \partial_x^3 f_1(x) - \partial_x(f_1^2(x))$. Then $v_j(x,t)$ for $j = 0,1,2$ satisfies

$$\begin{cases}
\partial_t^2 v_j - \partial_x^2 v_j + \partial_x^4 v_j + \partial_x^2 g_j = 0, & t \in [0,T'],\ 0 < x < 1, \\
v_j(0,t) = \partial_x^2 v_j(0,t) = 0, & v_j(1,t) = \partial_x^2 v_j(1,t) = 0, \\
v_j(x,0) = f_j(x), & \partial_t v_j(x,0) = \partial_x h_j(x),
\end{cases} \qquad (4.14)$$

where we set $g_0(x,t) = v_0^2(x,t)$, $g_1(x,t) = 2v_0v_1$ and $g_2(x,t) = 2v_0v_2 + 2v_1^2$. To prove the lemma, it is sufficient to prove that the solution $v_j(x,t)$ of (4.14) can be extended to $[0,T]$ for $j = 0$, 1 and 2.

It comes from Theorem 4.1 together with $(f_0,h_0) \in H_0^1(I) \times L^2(I)$ that the solution $v_0(x,t)$ to the IBVP (4.14) for $j = 0$ can be extended to $[0,T]$ and satisfies

$$\sup_{0\leqslant t\leqslant T}\big\|v_0(\cdot,t)\big\|_{H^1(I)} \leqslant C\big(\|h_0\|_{L^2(I)} + \|f_0\|_{H^1(I)}\big). \qquad (4.15)$$

Write, for $j = 1$ and 2,

$$v_j(x,t) = W_C(f_j,h_j)(x,t) + W_I(\partial_x g_j)(x,t).$$

It follows from Propositions 2.1 and 2.2 in Section 2 that, for $t \in [0,T']$,

$$\begin{aligned}
e^{-T}\big\|v_1(\cdot,t)\big\|_{H^1(I)} &\leqslant C\big(\|f_1\|_{H^1(I)} + \|h_1\|_{L^2(I)}\big) + C\int_0^t \big\|g_1(\cdot,\tau)\big\|_{H^1(I)}\,d\tau \\
&\leqslant C\big(\|f\|_{H_0^3(I)} + \|h\|_{H_0^2(I)}\big) + C\int_0^t \big\|v_0(\cdot,\tau)\big\|_{H^1(I)}\big\|v_1(\cdot,\tau)\big\|_{H^1(I)}\,d\tau \\
&\leqslant C\big(\|f\|_{H_0^3(I)} + \|h\|_{H_0^2(I)}\big) + \sup_{0\leqslant t\leqslant T}\big\|v_0(\cdot,t)\big\|_{H^1(I)}\int_0^t \big\|v_1(\cdot,\tau)\big\|_{H^1(I)}\,d\tau.
\end{aligned}$$

Using the Gronwall inequality together with (4.15) and the fact $v_1(0,t)=0$ and $v_1(1,t)=0$, we conclude that $v_1 \in C[0,T'];H_0^1(I))$ and for $t\in[0,T']$,

$$\big\|v_1(\cdot,t)\big\|_{H_0^1(I)} \leqslant C_T\big(\|f\|_{H_0^1(I)}+\|h\|_{L^2(I)}\big)\big(\|f\|_{H_0^3(I)}+\|h\|_{H_0^2(I)}\big), \tag{4.16}$$

where $C_T(\xi)$ is a T-dependent and non-decreasing function. Using Propositions 2.1 and 2.2 again, one obtains, for $t\in[0,T']$,

$$\begin{aligned} e^{-T}\big\|v_2(\cdot,t)\big\|_{H^1(I)} &\leqslant C\big(\|f_2\|_{H_0^1(I)}+\|h_2\|_{L^2(I)}\big)+C\int_0^t\big\|g_2(\cdot,\tau)\big\|_{H^1(I)}\,d\tau \\ &\leqslant C\big(\|f\|_{H_0^5(I)}+\|h\|_{H_0^4(I)}\big)+C\int_0^t\big\|v_1(\cdot,\tau)\big\|_{H^1(I)}^2\,d\tau+\sup_{0\leqslant t\leqslant T}\int_0^t\big\|v_2(\cdot,\tau)\big\|_{H^1(I)} \\ &\leqslant C\big(\|f\|_{H_0^5(I)}+\|h\|_{H_0^4(I)}\big)+C_1\int_0^t\big\|v_2(\cdot,\tau)\big\|_{H^1(I)}\,d\tau \\ &\quad+CT\big(C_T\big(\|f\|_{H_0^1(I)}+\|h\|_{L^2(I)}\big)\big(\|f\|_{H_0^3(I)}+\|h\|_{H_0^2(I)}\big)\big)^2. \end{aligned}$$

The Gagliardo–Nirenberg inequality means

$$\big(\|f\|_{H_0^3(I)}+\|h\|_{H_0^2(I)}\big)^2 \leqslant C\big(\|f\|_{H_0^1(I)}+\|h\|_{L^2(I)}\big)\big(\|f\|_{H_0^5(I)}+\|h\|_{H_0^4(I)}\big).$$

Then, based on (4.15) together with the Gronwall inequality and the fact $v_2(0,t)=0$ and $v_2(1,t)=0$, we conclude that for $t\in[0,T']$,

$$\big\|v_2(\cdot,t)\big\|_{H_0^1(I)} \leqslant D_T\big(\|f\|_{H_0^1(I)}+\|h\|_{L^2(I)}\big)\big(\|f_0\|_{H_0^5(I)}+\|h\|_{H_0^4(I)}\big), \tag{4.17}$$

where $D_T(\xi)$ is a T-dependent and non-decreasing function. Applying Lemma 4.1 to the IBVP (4.14) for $j=1$ yields, for $t\in[0,T']$,

$$\begin{aligned} &\int_0^1\left(\int_0^x\partial_t v_1(\eta,t)\,d\eta-\int_0^1(1-\eta)\partial_t v_1(\eta,t)\,d\eta\right)^2dx+\big\|v_1(\cdot,t)\big\|_{H_0^1(I)}^2-4\int_0^t\int_0^1 v_0v_1\partial_\tau v_1\,dx\,d\tau \\ &\quad\leqslant 2\|h_1\|_{L^2(I)}^2+\|f_1\|_{H_0^1(I)}^2. \end{aligned}$$

Because of $\partial_t v_0=v_1$, the Sobolev embedding theorem means

$$\begin{aligned} 2\left|\int_0^t\int_0^1 v_0v_1\partial_\tau v_1\,dx\,d\tau\right| &= \left|\int_0^t\int_0^1 v_0\partial_\tau v_1^2\,dx\,d\tau\right| \\ &\leqslant \int_0^1\big|v_0(x,t)\big|\big|v_1(x,t)\big|^2dx+\int_0^1\big|f(x)\big|\big|f_1(x)\big|^2dx+\left|\int_0^t\int_0^1 v_1^3(x,\tau)\,dx\,d\tau\right| \\ &\leqslant C\left(\big\|v_0(\cdot,t)\big\|_{H_0^1(I)}\big\|v_1(\cdot,t)\big\|_{L^2(I)}^2+\|f\|_{H_0^1(I)}\|f_1\|_{L^2(I)}^2\,dx+\int_0^t\big\|v_1(\cdot,\tau)\big\|_{H^1(I)}^3\,d\tau\right). \end{aligned}$$

Then we arrive at, by using (4.15)–(4.17),

$$\sup_{0\leqslant t\leqslant T'}\int_0^1\left(\int_0^x\partial_t v_1(\eta,t)\,d\eta-\int_0^1(1-\eta)\partial_t v_1(\eta,t)\,d\eta\right)^2dx\leqslant C_4, \tag{4.18}$$

where C_4 is a constant dependent of T and $\|f\|_{H_0^5(I)}+\|h\|_{H_0^4(I)}$. Similarly, applying Lemma 4.1 to the IBVP (4.14) for $j=2$ we get

$$\sup_{0\leqslant t\leqslant T'}\int_0^1\left(\int_0^x \partial_t v_2(\eta,t)\,d\eta-\int_0^1(1-\eta)\partial_t v_2(\eta,t)\,d\eta\right)^2 dx\leqslant C_4. \tag{4.19}$$

As in the proof of Theorem 4.1, that v_1 and v_2 can be extended to $[0,T]$, it follows from the estimates (4.16)–(4.19). Since the right sides of the inequalities (4.16) and (4.17) are independent of T' with $T'\leqslant T$, the estimates (4.16) and (4.17) hold also for all $t\in[0,T]$. Then the result in this lemma follows from applying (4.15) and (4.17) together with the Gagliardo–Nirenberg inequality to the equation

$$\partial_x^4 u=-v_2+\partial_x^2 u-\partial_x^2 u^2.$$

The proof is completed. □

Theorem 4.2. *For $T>0$ and $1\leqslant s\leqslant 5$, there exists a positive constant C such that, for any given $(f,h)\in H_0^s(I)\times H_0^{s-1}(I)$ with $\|f\|_{H_0^1(I)}+\|h\|_{L^2(I)}\leqslant C$, the IBVP* (1.1) *has a unique solution $u\in C([0,T];H^s(I))$.*

Proof. Choose a smooth function $0\leqslant\varphi(\xi)\leqslant 1$ satisfying $\varphi(\xi)=1$ for $|\xi|\leqslant\frac{C_0}{2}$ and $\varphi(\xi)=0$ for $|\xi|\geqslant C_0$, where C_0 is the constant obtained in Theorem 4.1. Consider the following IBVP with homogenous boundary conditions:

$$\begin{cases}\partial_t^2 u-\partial_x^2 u+\partial_x^4 u+\partial_x^2 u^2=0, & t>0,\ 0<x<1,\\ u(0,t)=\partial_x^2 u(0,t)=0, & u(1,t)=\partial_x^2 u(1,t)=0,\\ u(x,0)=\tilde f(x), & \partial_t u(x,0)=\partial_x\tilde h(x),\end{cases} \tag{4.20}$$

with

$$\tilde f(x)=\varphi\big(\|f\|_{H_0^1(I)}+\|h\|_{L^2(I)}\big)f(x),\qquad \tilde h(x)=\varphi\big(\|f\|_{H_0^1(I)}+\|h\|_{L^2(I)}\big)h(x).$$

To prove the theorem, it is sufficient to prove that, for any $f\in H_0^s(I)$ and $h\in H_0^{s-1}(I)$, the IBVP (4.20) admits a unique solution $u\in C([0,T];H^s(I))$.

It follows from Theorem 4.1 that for $(f,h)\in H_0^1(I)\times L^2(I)$, (4.20) possesses a unique solution $u\in C([0,T];H^1(I))$. Denote by $u=\mathcal{A}(f,h)$, then $\mathcal{A}$ is a nonlinear mapping from $H_0^1(I)\times L^2(I)$ into $C([0,T];H^1(I))$. It is obvious that Theorem 4.2 follows from the local well-posedness results established in Section 3 and the claim that $\mathcal{A}$ is a bounded mapping from $H_0^s(I)\times H_0^{s-1}(I)$ into $C([0,T];H^s(I))$.

To prove the claim, we shall use the nonlinear interpolation theory established in [5]. Choose

$$\mathcal{B}_0^1=H_0^1(I)\times L^2(I),\qquad \mathcal{B}_1^1=H_0^5(I)\times H_0^4(I),$$
$$\mathcal{B}_0^2=C\big([0,T];H^1(I)\big),\qquad \mathcal{B}_1^2=C\big([0,T];H^5(I)\big).$$

We conclude from Lemma 4.2 that, for any $(f_1,h_1),(f_2,h_2)\in\mathcal{B}_0^1$,

$$\begin{aligned}\big\|\mathcal{A}(f_1,h_1)-\mathcal{A}(f_2,h_2)\big\|_{\mathcal{B}_0^2}&=\big\|\mathcal{A}(f_1,h_1)-\mathcal{A}(f_2,h_2)\big\|_{C([0,T];H^1(I))}\\ &\leqslant A_T\left(\sum_{j=1}^2\big(\|\tilde f_j\|_{H_0^1(I)}+\|\tilde h_j\|_{L^2(I)}\big)\right)\big(\|\tilde f_1-\tilde f_2\|_{H_0^1(I)}+\|\tilde h_1-\tilde h_2\|_{L_0^2(I)}\big)\\ &\leqslant A_T\left(\sum_{j=1}^2\big(\|f_j\|_{H_0^1(I)}+\|h_j\|_{L^2(I)}\big)\right)\big(\|f_1-f_2\|_{H_0^1(I)}+\|h_1-h_2\|_{L_0^2(I)}\big)\\ &=A_T\big(\big\|(f_1,h_1)\big\|_{\mathcal{B}_0^1}+\big\|(f_2,h_2)\big\|_{\mathcal{B}_0^1}\big)\big\|(f_1-f_2,h_1-h_2)\big\|_{\mathcal{B}_0^1}.\end{aligned} \tag{4.21}$$

That $\mathcal{A}(f,h)(x,t) \in C([0,T]; H^5(I))$ for $(f,h) \in H_0^5(I) \times H_0^4(I)$ it follows from Lemma 4.3. Moreover, one has

$$\begin{aligned} \|\mathcal{A}(f,h)\|_{\mathcal{B}_1^2} &= \|\mathcal{A}(f,h)\|_{C([0,T];H^5(I))} \\ &\leqslant B_T\big(\|\tilde f\|_{H_0^1(I)} + \|\tilde h\|_{L^2(I)}\big)\big(\|\tilde f\|_{H_0^5(I)} + \|\tilde h\|_{H_0^4(I)}\big) \\ &\leqslant B_T\big(\|f\|_{H_0^1(I)} + \|h\|_{L^2(I)}\big)\big(\|f\|_{H_0^5(I)} + \|h\|_{H_0^4(I)}\big) \\ &= B_T\big(\|(f,h)\|_{\mathcal{B}_0^1}\big)\|(f,h)\|_{\mathcal{B}_1^1}. \end{aligned} \tag{4.22}$$

Based on the estimates (4.21) and (4.22), the nonlinear interpolation theory (see [5, Theorem 1]) yields that, for $1 \leqslant s \leqslant 5$ and $(f,h) \in H_0^s(I) \times H_0^{s-1}(I)$, $\mathcal{A}(f,h) \in C([0,T]; H^s(I))$ and

$$\|\mathcal{A}(f,h)\|_{C([0,T];H^s(I))} = \|\mathcal{A}(f,h)\|_{\mathcal{B}_{\theta,2}^2} \leqslant C_1\|(f,h)\|_{\mathcal{B}_{\theta,2}^1} = C_1\big(\|f\|_{H_0^s(I)} + \|h\|_{H_0^{s-1}(I)}\big), \tag{4.23}$$

where C_1 is a constant depending only on s, T and $\|f\|_{H_0^1(I)} + \|h\|_{L^2(I)}$, $\theta = \frac{s-1}{4}$. The claim is proved. □

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